



TITLE:

# A Hierarchy of the Fragments of the System of Inductive Definition : Preliminary Report

AUTHOR(S):

Hamano, Masahiro; Okada, Mitsuhiro

---

CITATION:

Hamano, Masahiro ...[et al]. A Hierarchy of the Fragments of the System of Inductive Definition : Preliminary Report. 数理解析研究所講究録 1997, 976: 169-181

ISSUE DATE:

1997-02

URL:

<http://hdl.handle.net/2433/60797>

RIGHT:

# A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

Masahiro Hamano\* and Mitsuhiro Okada†

Department of Philosophy  
Keio University, Tokyo

## 1 introduction

Gentzen [7] proved the consistency of  $PA$  (Peano Arithmetic) by using the transfinite induction up to the first *epsilon* number  $\epsilon_0$ . Here  $\epsilon_0$  is  $\lim_k \omega_k$ , where  $\omega_0 = 0$  and  $\omega_{k+1} = \omega^{\omega_k}$ . Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than  $\epsilon_0$ , eg.,  $\omega_k$  for any natural number  $k$ , is provable in  $PA$ .

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher  $\omega$ -tower  $\omega_{k+1}$  is proved from the accessibility of  $\omega_k$ . Hence by considering Gentzen's work [7, 8] a natural question arises; does the hierarchy of  $\omega$ -towers,  $\{\omega_k\}_{k=1,2,\dots}$ , correspond exactly to a certain hierarchy of fragments of  $PA$ ?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of  $PA$ , where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of  $\xi$ -iterated Inductive Definition  $ID_\xi$  [6]. We first analyze in Section 2 Arai's optimal accessibility proof for  $ID_\xi$  ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic  $ID_\xi$ , where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of  $ID_\xi$  ([3]). In fact, for the upper bounds proof we use the fragments of classical  $ID_\xi$  in terms of the nestedness complexity of classical negations. Since the fragments of  $ID_\xi$  obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier  $\exists$  by means of  $\neg\forall\neg$ ), our result for  $ID_\xi$  corresponds to Mints' ([10]) for  $PA$ .

\*慶応義塾大学哲学科 日本学術振興会特別研究員 浜野 正浩 hamano@abelard.flet.mita.keio.ac.jp

†慶応義塾大学哲学科 岡田 光弘 mitsu@abelard.flet.mita.keio.ac.jp

## 2 Provability of transfinite inductions on $\omega(\xi, k, 0)$ in subsystems of $S_k(ID_\xi^i(\mathcal{U}_0))$

Let  $(I, \prec)$  be the well ordered system whose order type is ordinal  $\xi + 1$ . Arai [1] proved the well ordering of Takeuti's system of ordinal diagram  $O(\xi + 1, 1)$  in the system  $ID_\xi^i$  (the intuitionistic system of  $\xi$ -times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments  $S_k(ID_\xi^i)$  of  $ID_\xi^i$  based on the nestedness complexity of implications, and observe Arai's well ordering proof of [1] on these fragments.

Now we recall the definitions of  $ID_\xi^i(\mathcal{U})$  and  $ID_\xi^i$  of Feferman [6].

**Definition 1** (System  $ID_\xi^i(\mathcal{U})$  and  $ID_\xi^i$ , cf. Feferman [6])

For any positive operator form  $\mathcal{U}$ ,  $ID_\xi^i(\mathcal{U})$  is obtained from PA by adding the following axiom schemata.

$$(P_\xi.1) \quad \rightarrow \forall x \prec \xi (A(P_x^\mathcal{U}, P_{\prec x}^\mathcal{U}, x) \subseteq P_x^\mathcal{U})$$

$$(P_\xi.2) \quad \rightarrow \forall x \prec \xi (\mathcal{U}(V, P_{\prec x}^\mathcal{U}, x) \subseteq V \supset P_x^\mathcal{U} \subseteq V)$$

$$(TI)_\xi \quad \text{Prog}[I, \prec, V] \rightarrow (I \subseteq V)$$

$$\text{where } P_{\prec a}^\mathcal{U} := \{x, y\} (x \prec a \wedge P^\mathcal{U} xy)$$

$$ID^i := \bigcup \{ID_\xi^i(\mathcal{U}) \mid \mathcal{U} \text{ is a positive operator form}\}$$

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to  $\prec_i$  for  $i \prec \xi$  (cf. §26 [14]) by using the set constants  $A_i$  which is definable in  $ID_\xi^i(\mathcal{U}_0)$  with the following  $\mathcal{U}_0$ ;

$$(A.1)_\xi \quad \forall i \prec \xi \text{Prog}[F_i, \prec_i, A_i]$$

$$(A.2)_\xi \quad \forall i \prec \xi (\text{Prog}[F_i, \prec_i, V] \rightarrow A_i \subseteq V) \quad \text{for each abstract } V \text{ in } ID_\xi^i(\mathcal{U}_0)$$

where  $\mathcal{U}_0$  is a  $X$ -positive operator form defined as  $\mathcal{U}_0(X, Y, i, \mu) := \mathcal{F}(i, \mu, Y) \wedge \forall \nu \prec_i \mu (\mathcal{F}(i, \nu, Y) \rightarrow X(\nu))$  where  $\mathcal{F}(i, \mu, Y) := \forall k \prec i \forall \rho \subseteq_k \mu Y(k, \rho)$ ,  $\text{Prog}[\alpha, \gamma, \beta] := \forall x (\alpha(x) \wedge \forall y (\gamma(y, x) \wedge \alpha(y) \rightarrow \beta(y)) \rightarrow \beta(x))$ , and  $F_i(\mu) := \forall j \prec i \forall \nu \subseteq_j \mu A_j(\nu)$  (the intended meaning of  $F_i(\mu)$  is that  $\mu$  is an  $i$ -fan (cf. Definition 26.16 [14])).

Remember that  $ID_\xi^i(\mathcal{U})$  has the mathematical induction of the following form;

$$(VJ) \quad V(0), \forall x (V(x) \rightarrow V(x')) \rightarrow V(t)$$

The above  $ID_\xi^i(\mathcal{U}_0)$  is the specific subsystem of the system  $ID_\xi^i$  of Inductive Definition in which the induction schemata are used only for the accessibility predicate  $A_i$  of ordinals.

We consider the subsystem  $S_k(ID_\xi^i(\mathcal{U}_0))$  of  $ID_\xi^i(\mathcal{U}_0)$  where each abstract  $V$  in  $(A.2)_\xi$ ,  $(TI)_\xi$  and  $(VJ)$  is restricted to that of level  $lv(V) \leq k$ ; where  $lv(V)$  is defined by the definition below.

We introduce the notion of level of  $A$  ( $lv(A)$ ) for a formula  $A$  to express, roughly speaking, the implicational complexity of  $A$ . We assume that the language contains only  $\forall$ ,  $\supset$  and  $\wedge$  for the logical connectives in this section.

We first recall the degree  $d$  of a formula in the language of  $ID_\xi^i(\mathcal{U})$  defined in Arai [3], which intends to indicate how many times inductive definition is applied.

**Definition 2** (cf. Def 2.4 in Arai [3])

- $d(t = s) = 0$  for all term  $t, s$  and predicate variable  $X$ .

•

$$d(P^\mathcal{U} ts) = \begin{cases} i \oplus 1 & \text{if } t \text{ is a closed term whose value is } i \prec \xi. \\ \xi & \text{otherwise} \end{cases}$$

$$d(t_1 \prec s \wedge P^{\mathcal{U}} t_2 r) = \begin{cases} i & \text{if } s \text{ is a closed term whose value is } i \prec \xi \text{ and } t_1 \text{ is a} \\ & \text{closed term representing the same numeral as } t_2. \\ \xi & \text{otherwise} \end{cases}$$

**Definition 3** (level  $lv(A)$  of formula  $A$  in the language of  $ID_{\xi}^i(\mathcal{U})$ ) For the formula  $A$  in the language of  $ID_{\xi}^i(\mathcal{U})$ , the level  $lv(A)$  of the formula  $A$  is defined inductively as follows:

$$\begin{aligned} lv(P) &:= 0 \text{ for any atom of the language of } PA. \\ lv(A \wedge B) &:= \max\{lv(A), lv(B)\} \\ lv(\forall x A) &:= \begin{cases} \max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \\ lv(A \supset B) &:= \begin{cases} \max\{lv(A) + 1, lv(B)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \\ lv(P^{\mathcal{U}} t) &:= \begin{cases} 1 & \text{if } d(P^{\mathcal{U}} t) = \xi \\ 0 & \text{otherwise} \end{cases} \\ lv(t \prec s \wedge P_t^{\mathcal{U}}) &:= \begin{cases} 1 & \text{if } d(P_t^{\mathcal{U}}) = \xi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The subsystems  $S_k(ID_{\xi}^i(\mathcal{U}))$  and  $S_k(ID_{\xi}^i)$  of  $ID_{\xi}^i(\mathcal{U})$  and  $ID_{\xi}^i$  are defined in terms of level  $lv$  as follows;

**Definition 4** (the subsystem  $S_k(ID_{\xi}^i(\mathcal{U}))$  of  $ID_{\xi}^i(\mathcal{U})$ )  $S_k(ID_{\xi}^i(\mathcal{U}))$  is  $ID_{\xi}^i(\mathcal{U})$  except that for every abstract  $V$  in  $(A.2)_{\xi}$ ,  $(TI)_{\xi}$  and  $(VJ)$ ,  $lv(V) \leq k$  holds.  
 $S_k(ID_{\xi}^i) := \bigcup \{S_k(ID_{\xi}^i(\mathcal{U})) \mid \mathcal{U} \text{ is a positive operator form}\}$

The following notation is introduced;

**Notation 1** Let  $TI[\alpha, \gamma, \mu]$  denote the schema defined as  $TI[\alpha, \gamma, \mu] := \alpha(\mu) \wedge (Prog[\alpha, \gamma, V] \rightarrow \forall \nu (\gamma(\mu, \nu) \wedge \alpha(\nu) \rightarrow V(\nu)))$ . And  $TI[\alpha, \gamma, \mu]_Q$  is the result of  $TI[\alpha, \gamma, \mu]$  by substituting  $Q$  for  $V$

**Notation 2**  $\omega(\xi, 0, \alpha) := \alpha$  and  $\omega(\xi, n+1, \alpha) := (\xi, \omega(\xi, n, \alpha))$ .

Then by checking Arai's well ordering proof of  $O(\xi+1, 1)$  [1] carefully, Proposition 1 is easily observed.

**Proposition 1** For a formula  $Q$  with  $lv(Q) \leq 2$  and  $k > 2$ ,  $TI[F_0, <_0, \omega(\xi, k, 0)]$  is provable in  $S_k(ID_{\xi}^i(\mathcal{U}_0))$ . Namely, the ordinal  $\omega(\xi, k, 0)$  is accessible in  $S_k(ID_{\xi}^i(\mathcal{U}_0))$  with respect to  $<_0$ .

**Proof.**

We follow Arai's [1].

We only consider the case in which  $\xi$  is a limit. (See Remark after Proposition 2 for the successor  $\xi$  case.) Let  $\bigcap_{k \prec i} A_k := \{\mu\} \forall k \prec i A_k(\mu)$ . In Lemma 3 of [1]  $(TI)_{\xi}$  is used with the abstract  $\{i\} Prog[F_i, <_i, \bigcap_{k \prec i} A_k] := \{i\} \forall x (F_i(x) \wedge \forall y <_i x (F_i(y) \rightarrow \bigcap_{k \prec i} A_k(y)) \rightarrow \bigcap_{k \prec i} A_k(x))$ , here  $lv(Prog[F_i, <_i, \bigcap_{k \prec i} A_k(\mu)]) = 3$ . Let  $\bar{A} := \bigcap_{j \prec \xi} A_j$  and  $R_i(\nu) := \forall \mu <_{\xi} (i, \nu) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$ . In Lemma 4 of [1]  $(A.2)_{\xi}$  is used with the abstract  $\{x\} R_i(x) := \forall \mu <_{\xi} (i, x) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$  (with  $lv(R_i(x)) = 2$ ) and  $(TI)_{\xi}$  is used with the abstract  $\{i\} R_i(0) := \forall \mu <_{\xi} (i, 0) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$  (with  $lv(R_i(0)) = 2$ ).

Then in Lemma 5 of [1] it is shown that  $TI[F_{\xi}, <_{\xi}, (\xi, 0)]_Q$  is provable in  $ID_{\xi}^i(\mathcal{U}_0)$  for each unary predicate  $Q(x)$  in  $ID_{\xi}^i(\mathcal{U})$ ; In the case where  $\lim(\xi)$ ,  $(A.2)_{\xi}$  are used

with the abstract  $\{x\}(x \prec_\xi (i, 0) \rightarrow Q(x))$  for all  $i \prec \xi$  (with level  $lv(Q)$ ). In the case where  $Suc(\xi)$ ,  $(A.2)_\xi$  is used with the abstract  $\{x\}(x <_\xi (\xi, 0) \rightarrow Q(x))$  (with level  $lv(Q)$ ).

Hence until now it is observed that

$$(I) \quad S_{Max(3, lv(Q))}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, (\xi, 0)]_Q.$$

From (I) it is derived in the way familiar by Gentzen [8] that

$$(II) \quad S_{k+3}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, k+3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.$$

Let us observe the proof of (II). In Lemma 7 of [1] it is shown that  $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$ , where  $s[Q]$  is a jump operator defined as  $s[Q](\mu) := \forall \rho (F_\xi(\rho) \rightarrow \forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <_\xi \rho + (\xi, \mu)^\xi (F_\xi(\nu) \rightarrow Q(\nu)))$ , where  $\lambda \nu \mu. \mu + \nu^\xi$  is a primitive recursive function which is a generalization of  $\lambda \nu \mu. \nu + \omega^\mu$  of Gentzen [8] and defined in [1] as follows;

- If  $\mu = 0$ , then  $\mu + \nu^\xi = \nu + \mu^\xi = \nu$
- Suppose  $\mu \neq 0$  and  $\nu \neq 0$  and  
 $\mu \equiv \mu_1 \# \dots \# \mu_m$  with  $\mu_1 \geq_\xi \dots \geq_\xi \mu_m \neq 0$   
 $\nu \equiv \nu_1 \# \dots \# \nu_n$  with  $\nu_1 \geq_\xi \dots \geq_\xi \nu_n \neq 0$   
 Let  $l$  be the number such that  $0 \leq l \leq m$  and  $\mu_l \leq_\xi \nu_1 <_\xi \mu_{l+1}$ ,  
 then  $\mu + \nu^\xi := \mu_1 \# \dots \# \mu_l \# \nu_1 \# \dots \# \nu_n$

Note that  $lv(s^n[Q]) = n + Max(2, lv(Q))$  with  $n \geq 1$ , where  $s^n[Q] := \overbrace{s[\dots s[Q]\dots]}^{n\text{-times}}$ .

Let us sketch the proof of  $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$  due to Gentzen [8], where a mathematical induction of the level  $\leq lv(Q)$  is used;

Assume

$$Prog[F_\xi, <_\xi, Q] \quad \dots (1)$$

$$F_\xi(x) \wedge \forall y <_\xi x (F_\xi(y) \rightarrow s[Q](y)) \quad \dots (2)$$

We have to show  $s[Q](x)$ . So assume further

$$F_\xi(\rho) \quad \dots (3)$$

$$\forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \quad \dots (4)$$

$$\nu <_\xi \rho \oplus (\xi, x)^\xi \wedge F_\xi(\nu) \quad \dots (5)$$

Under the above assumptions (1)  $\sim$  (5), we have to show  $Q(\nu)$ .

Consider the case where  $x \neq 0$ . Since  $\nu <_\xi \rho \oplus (\xi, x)^\xi$ , there exists primitive recursive functions  $f$  and  $g$  such that  $\nu <_\xi \rho \oplus (\xi, f(x, \nu, \rho)) \cdot g(x, \nu, \rho)$  with  $f(x, \nu, \rho) <_\xi x$  and  $F_\xi(f(x, \nu, \rho))$ . From (2),  $s[Q](f(x, \nu, \rho))$  holds. Then a universal instantiation with  $\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n$  (note that  $\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n <_\xi \rho \oplus (\xi, x)^\xi$ ) for an arbitrary  $n$  allows the following:

$$F_\xi(\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <_\xi (\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n) \oplus (\xi, f(x, \nu, \rho)^\xi) (F_\xi(\eta) \rightarrow Q(\eta)) \dots (6)$$

From  $F_\xi(\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n)$  (from (5)) and the property of  $Suc$ , the following holds;

$$\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot Suc(n) (F_\xi(\eta) \rightarrow Q(\eta)) \dots (7)$$

Then mathematical induction with abstract  $\{n\}(\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)))$ , whose level is  $Max(2, lv(Q))$ , implies (with (4))  $\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot g(x, \nu, \rho) (F_\xi(\eta) \rightarrow Q(\eta))$ . Hence from (5),  $Q(\nu)$  holds.

Consider the case where  $x = 0$ . For each formula  $Q$ ,  $s[Q]$  denotes the formula of the following form;  $s[Q](\mu) := \forall \rho (F_\xi(\rho) \rightarrow \forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <_\xi \rho + \mu^\xi (F_\xi(\nu) \rightarrow Q(\nu)))$ . Then we can prove without  $(A.1)_\xi$ ,  $(A.2)_\xi$ ,  $TI_\xi$  and the mathematical induction that  $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$ . As is shown

above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level  $\leq \text{Max}(2, \text{lv}(Q))$ .

From now we assume  $\text{lv}(Q) \leq 2$ . With the help of  $\text{Prog}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}[F_\xi, <_\xi, s[Q]]$  and  $\text{Prog}[F_\xi, <_\xi, s[Q]] \rightarrow \text{Prog}[F_\xi, <_\xi, s^2[Q]]$ , in which proof all mathematical inductions are restricted to those of level  $\leq 3$ , (I) implies the following (II)<sub>0</sub>;

$$(II)_0 \quad S_3(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, 3, 0)]_Q$$

By replying this methode, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

$$S_{k+3}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_0, <_0, \omega(\xi, k+3, 0)]_Q \text{ with } \text{lv}(Q) \leq 2 \text{ and } k \geq 0.$$

Hence the proposition holds.

□

Using the above, Proposition 2 follows;

**Proposition 2** *For  $k > 2$ , the ordinal up to  $\omega(\xi, k+1, 0)$  is accessible in  $S_k(ID_\xi^i(\mathcal{U}_0))$  with respect to  $<_0$ .*

Remark 1;

From the case in which  $\xi$  is a successor ordinal, the transfinite induction formula  $\{i\}\text{Prog}[F_i, <_i, \bigcap_{k < i} A_k]$  at the beginning of the proof of Proposition 1 above is replaced by  $\{i\}\text{Prog}[F_i, <_i, A_i]$ , which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for  $k > 1$ .

### 3 Unprovability of the transfinite induction up to $\omega(\xi, k+1, 0)$ in system $S_k(AI_\xi^-)$

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

$$S_k(ID_\xi) \not\vdash TI[F_0, <_0, \omega(\xi, k+1, 0)] \text{ for } k > 2$$

On the whole segment of  $ID_\xi = \bigcup_n S_n(ID_\xi)$ , Arai [3] proves that  $ID_\xi \not\vdash TI[F_0, <_0, O(\xi+1, 1)]$ . Note that  $O(\xi+1, 1) := \bigcup_k \omega(\xi, k, 0)$ . He shows that the consistency of  $ID_\xi$  is provable using tranfinite induction up to  $O(\xi+1, 1)$  by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

$$TI[F_0, <_0, \omega(\xi, k+1, 0)] \vdash \text{Cons}(S_k(ID_\xi)) \text{ for } k > 2$$

Our crucial point is to introduce a  $\eta$ -height  $h_\eta$  for each  $\eta \preceq \xi$  (Definition 11) and consider a ordinal assignment to a proof  $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$  with  $\xi$ -sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system  $AI_\xi^-$  of  $\xi$ -times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System  $AI_\xi^-$  is defined by adding the following principles based on  $PA$ .

**Definition 5** (System  $AI_{\xi}^{-}$ , cf. Arai [3])

For any arithmetical form  $\mathcal{B}$ , the following axioms schemata are added.

$$(Q^{\mathcal{B}} : right) \quad \frac{\Gamma \rightarrow \Delta, \mathcal{B}(X, Q_{\prec t}^{\mathcal{B}}, t, s)}{\Gamma \rightarrow \Delta, Q^{\mathcal{B}}ts} \quad \text{where } Q_{\prec t}^{\mathcal{B}} := \{x, y\}(x \prec t \wedge Q^{\mathcal{B}}xy)$$

$$(Q^{\mathcal{B}} : left) \quad t \prec \xi, Q^{\mathcal{B}}ts \rightarrow \mathcal{B}(V, Q_{\prec t}^{\mathcal{B}}, t, s)$$

We assume that the language contains only  $\forall, \neg$  and  $\wedge$  for the logical connectives. Then, the definition of  $lv$  in the previous section is modified as follows;

**Definition 6** ( $\eta$ -level  $lv_{\eta}(A)$  of a formula  $A$  with  $\eta \preceq \xi$ ) For the formula  $A$  in the language of  $AI_{\xi}^{-}$  and an ordinal  $\eta \preceq \xi$ , the  $\eta$ -level  $lv_{\eta}(A)$  of the formula  $A$  is defined inductively as follows, where  $d$  is defined in Definition 2 of previous section with using  $Q^{\mathcal{B}}$  instead of  $P^{\mathcal{U}}$  and  $d(Xt) := 0$  (for  $X$  a predicate variable):

$$\begin{aligned} lv_{\eta}(P) &:= 0 \text{ for any atom of } L_{PA}. \\ lv_{\eta}(A \wedge B) &:= \max\{lv_{\eta}(A), lv_{\eta}(B)\} \\ lv_{\eta}(\forall x A) &:= \begin{cases} \max\{2, lv_{\eta}(A)\} & \text{if } lv_{\eta}(A) \geq 1 \\ 0 & \text{if } lv_{\eta}(A) = 0 \end{cases} \\ lv_{\eta}(\neg A) &:= \begin{cases} lv_{\eta}(A) + 1 & \text{if } lv_{\eta}(A) \geq 1 \\ 0 & \text{if } lv_{\eta}(A) = 0 \end{cases} \\ lv_{\eta}(Q_t^{\mathcal{B}}) &:= \begin{cases} 1 & \text{if } d(Q_t^{\mathcal{B}}) = \eta \\ 0 & \text{otherwise} \end{cases} \\ lv_{\eta}(t \prec s \wedge Q_t^{\mathcal{B}}) &:= \begin{cases} 1 & \text{if } d(t \prec s \wedge Q_t^{\mathcal{B}}) = \eta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that  $lv_{\eta}$  for  $\eta = \xi$  is the same as  $lv$  of the previous section (with using  $Q^{\mathcal{B}}$  instead of  $P^{\mathcal{U}}$  in the definition of  $lv$  of the previous section with replacing  $\supset$  by  $\neg$ .)

We can define the fragments  $S_k(AI_{\xi}^{-})$  in the same manner as  $S_k(ID_{\xi})$  as follows.

**Definition 7** (the subsystem  $S_k(AI_{\xi}^{-})$  of  $AI_{\xi}^{-}$ )  $S_k(AI_{\xi}^{-})$  is  $AI_{\xi}^{-}$  except that for every abstract  $V$  in  $Q^{\mathcal{B}}:left$  and  $(VJ)$ ,  $lv_{\xi}(V) \leq k$  holds.

$ID_{\xi}$  is obtained from  $ID_{\xi}^i$  in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula  $F$  of the language of  $ID_{\xi}$ , we define a formula  $F^*$  of the language of  $AI_{\xi}$  by substituting  $Q^{\mathcal{B}}$  for all occurrences of  $P^{\mathcal{U}}$ , where

$$\mathcal{B}(X, Y, c_0, c_1) := \forall y(U(X, Y, c_0, y) \rightarrow Xy) \rightarrow Xc_1.$$

It is well known that by this  $*$ ,  $ID_{\xi}$  is embeddable into  $AI_{\xi}$  (cf. [3]). Obviously  $lv(F) = lv_{\xi}(F^*)$  holds i.e.,  $\xi$ -level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occurring in a proof figure of  $AI_{\xi}^{-}$  are of the following normal form:

**Lemma 1** (the normal form of a formula in  $AI_{\xi}^{-}$ ) For arbitrary formula  $A$  of the language of  $AI_{\xi}^{-}$ , there exists a formula of the following form, called a normal formula, which is equivalent to  $A$  (in  $LK$ );

$$\forall \vec{x}_1 \neg \dots \forall \vec{x}_n \forall \neg \quad \forall \vec{y} D[Q^{\mathcal{B}}_{t_1 s_1}, \dots, Q^{\mathcal{B}}_{t_m s_m}]$$

where  $D[*_1, \dots, *_m]$  is a context of the language of  $PA$ , and no quantifier occurring in  $D$  bounds any  $*_i$  ( $1 \leq i \leq m$ ) and  $lv_{\eta}(D[Q^{\mathcal{B}}_{t_1 s_1}, \dots, Q^{\mathcal{B}}_{t_m s_m}]) \leq 2$  for any  $\eta \preceq \xi$ .

**Definition 8 (normal proofs)** Let  $S$  be a sequent of normal formulas. A normal proof of  $S$  is a proof in which  $\forall\neg$ -left rules are used, instead of  $\forall$ -left rules in a proof;

$$\frac{\Gamma \rightarrow \Delta, A(t_1, \dots, t_n)}{\forall x_1 \dots x_n \neg A(x_1, \dots, x_n), \Gamma \rightarrow \Delta} \forall\neg\text{-left}$$

Note that the original  $\neg$ -left rule may also appear in a normal proof.

**Lemma 2** Any provable sequent of normal formulas has a normal proof.

From now on we assume any  $S_k(Al_\xi^-)$ -proof to be normal by virtue of the above two lemmata.

**Definition 9** For each formula  $A$ ,  $\eta(A) \preceq \xi$  is defined as  $\eta(A) := \text{Max}\{\eta \mid lv_\eta(A) \neq 0\}$ .

**Definition 10** ( $g_\eta(A)$  with  $\eta \prec \xi$ )

$$g_\eta(A) := \begin{cases} g(A) & \text{if } \eta(A) \geq \eta \\ 0 & \text{if } \eta(A) < \eta \end{cases}$$

where  $g(A)$  denotes the number of logical symbols in  $A$ .

We modify the notion of proof with degree  $\langle P, d \rangle$  of Arai [3] into  $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$  by introducing  $\xi$ -sort of height  $\{h_\eta\}_{\eta \preceq \xi}$ , as follows:

**Definition 11** (A proof with  $\xi$ -sort of height  $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ ) A proof  $\langle P, d \rangle$  (with degree  $d$ ) is called a proof with  $\xi$ -sort of height  $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$  if for each sequent  $S$  of  $P$  and each ordinal  $\eta \preceq \xi$ , a natural number  $h_\eta(S)$  satisfying the following condition is assigned. We call  $h_\eta$  a  $\eta$ -height.

0.  $h_\eta(S) = 0$  for every  $\eta \preceq \xi$  if  $S$  is the end sequent of  $P$ .

For the last inference  $I$  of the form

$$I \quad \frac{S}{S'}$$

1.  $h_\eta(S) = 0$  for every  $\eta \preceq \xi$  if  $I$  is a substitution.
2.  $h_\eta(S) = h_\eta(S')$  for every  $\eta \preceq \xi$  if  $I$  is an inference except substitution, induction and cut.
3. 
$$\begin{cases} 1 & h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} \text{ for } \eta \prec \xi \\ 2 & h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} \end{cases}$$
 if  $I$  is a cut, where  $D$  is the cut formula of the inference  $I$ .
4. 
$$\begin{cases} 1 & h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} + 1 \text{ for } \eta \prec \xi \\ 2 & h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} + 1 \end{cases}$$
 if  $I$  is an induction.

**Definition 12** For each sequent  $S$  of  $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ ,  $\eta(S) \preceq \xi$  is defined as  $\eta(S) := \begin{cases} d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\ \text{Max}\{\eta \mid h_\eta(S) \neq 0\} & \text{otherwise} \end{cases}$

The following is an immediate consequence from Definition 12.



**Lemma 3** For any proof with  $\xi$ -sort of height  $< P, \{h_\eta\}_{\eta \leq \xi}, d >$  and for any inference  $I$  (with a lower sequent  $S'$  and an upper sequent  $S$ ) in  $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ ,

$$\eta(S) \geq \eta(S')$$

holds.

**Notation 3** For  $i \leq \xi$  and an ordinal diagram  $\alpha$ , an ordinal diagram  $\omega(i, n, \alpha)$  is defined inductively as follows.

- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha))$

**Definition 13** (ordinal assignment) Let  $I$  be an inference of the form

$$I \quad \frac{S_1 \quad S_2}{S}$$

Then  $O(S)$  is defined as follows:

1. When  $I$  is a cut,

$$O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), c[\omega(\eta(S_1), h_{\eta(S_1)}(S_1), O(S_1) \# O(S_2))])$$

Here  $k := \text{Max}\{h_{\eta(S)}(T) \mid T \text{ is above } I\}$  and  $c[*] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \dots, \omega(\gamma_n, k_n, *)))$ , where  $\{\gamma_1, \dots, \gamma_n\} := \{\gamma \mid \eta(S) < \gamma < \eta(S_1) \text{ and } h_\gamma(T) \neq 0 \text{ for some } T \text{ above } I\}$  with  $\gamma_1 < \dots < \gamma_n$  and  $k_i := \text{Max}\{h_{\gamma_i}(T) \mid T \text{ is above } I\}$ .<sup>1</sup>

2. When  $I$  is a logical inference,

$$O(S) := O(S_1) \# O(S_2) \# 0$$

3. When  $I$  is a structural inference,

$$O(S) := O(S_1) \# O(S_2)$$

4. When  $I$  is a substitution,

$$O(S) := (d(I), O(S_1))$$

**Theorem 1** The transfinite induction on  $\omega(\xi, k + 1, 0)$  is unprovable in  $S_k(AI_\xi^-)$  for  $k > 2$ .

**Proof.**

We refine the proof reduction process of Arai [3] to define the reduction process for  $S_k(AI_\xi^-)$  ( $k > 2$ ), and show that the well-orderness of  $\omega(\xi, k + 1, 0)$  implies the termination of the reduction process, hence the consistency of  $S_k(AI_\xi^-)$ . Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)

Without loss of generality, we assume that all logical initial sequents of the form  $p \rightarrow p$  where  $p$  is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) does work not only for a weakening in end-piece but also for a more general weakening with such a weakening formula  $D$  as the bundle  $I$  (cf. p78 of [14]) which begins with  $D$  ends with a cut formula  $D$  and no logical inference affect  $I$ .

<sup>1</sup>In the case where  $\eta(S) = \eta(S_1)$ ,  $c[*]$  is  $*$  and  $O(S) := \omega(\eta(S), k + h_{\eta(S)}(S_1) - h_{\eta(S)}(S), O(S_1) \# O(S_2))$ .

(elimination of the mathematical induction rule) As usual.

Then from sublemma 12.9 of [14], there exists a suitable cut  $J$  in the end piece of  $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$ . Let  $I_1$  and  $I_2$  be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of  $J$ .

We shall demonstrate following three essential cases both for limit ordinal  $\xi$  and for successor ordinal  $\xi$ ;

(Case 1) The case where the cut formula  $C := A \wedge B$  with  $\eta(C) \prec \xi$ :

Let  $K$  (whose lower sequent is  $T$  and whose upper sequent is  $T_1$ ) denotes the uppermost inference below  $J$  such that either (i) or (ii) holds;

$$\begin{aligned} \eta(T) &= \eta(A) \wedge (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i) \\ \eta(T) &< \eta(A) \cdots (ii) \end{aligned}$$

where  $A$  is the auxiliary formula of  $I_1$  and  $I_2$

$\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$  is as follows:

$$\begin{array}{c} \frac{\frac{S_1^{I_1}}{S_1^{I_1}} I_1 \quad \frac{S_2^{I_2}}{S_2^{I_2}} I_2}{S_1^J \quad S_2^J} J \\ \frac{T_1}{T} K \\ \vdots \end{array} \quad \begin{array}{l} S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, A_1 \\ S_2^{I_2} : \Gamma_2 \rightarrow \Delta_2, B_1 \\ S_1^{I_1} : \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A_1 \wedge B_1 \\ S_1^{I_2} : A_3, \Pi_3 \rightarrow \Lambda_3 \\ S_2^{I_2} : A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\ S_1^J : \Gamma \rightarrow \Delta, A \wedge B \\ S_2^J : A \wedge B, \Pi \rightarrow \Lambda \\ S_1^J : \Gamma, \Pi \rightarrow \Delta, \Lambda \\ T : \Phi \rightarrow \Psi \end{array}$$

$\langle P', \{h'_\eta\}_{\eta \leq \xi}, d' \rangle$  is as follows, where  $\tilde{I}_1$  and  $\tilde{I}_2$  are weakening-right (with a weakening formula  $A_1$ ) and weakening-left (with a weakening formula  $A_3$ ) respectively;

$$\begin{array}{c} \frac{\frac{S_1^{I_1}}{S_1^{I_1}} \tilde{I}_1 \quad \frac{S_2^{I_2}}{S_2^{I_2}} \tilde{I}_2}{S_1^* \quad S_2^*} J \\ \frac{T_1^*}{U_1} K \quad \frac{T_2^*}{U_2} K \\ \frac{T^*}{I} \end{array} \quad \begin{array}{l} S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, A_1, A_1 \wedge B_1 \\ S_1^* : \Gamma \rightarrow \Delta, A, A \wedge B \\ S_2^{I_2} : A_3, A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\ S_2^* : A \wedge B, A, \Pi \rightarrow \Lambda \\ U_1 : \Phi \rightarrow \Psi, A \\ U_2 : A, \Phi \rightarrow \Psi \\ T^* : \Phi, \Phi \rightarrow \Psi, \Psi \end{array}$$

(case 1.1): The case where (i) holds. Then for any sequent  $T'$  between  $S_1$  and  $T$ ,  $\eta(T') \geq \eta(A)$  holds.

(case 1.1.1)  $\eta(T_1) = \eta(T)$

$O_{P'}(T^*) <_0 O_P(T)$  is checked as usual way.

(case 1.1.2)  $\eta(T_1) > \eta(T)$

special case of (case 1.2)

(Case 1.2): The case where (ii) holds. Then  $\eta(T) < \eta(A) \leq \eta(T_1)$  holds. We assign

$$h'_\eta(U_1) := \begin{cases} h_\eta(T) & \text{if } \eta < \eta(T) \\ g(A) & \text{if } \eta = \eta(A) \text{ , and } h'_\eta(T_1^*) := h'_\eta(T_1^{**}) := h_\eta(T_1) \text{ for all } \eta \leq \xi. \\ 0 & \text{otherwise} \end{cases}$$

Hence  $\eta(U_1) = \eta(A)$  holds. On the other hand, there exist contexts  $a$  and  $b$  such

that  $O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), a[\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2])])$ ,  
 $O_{P'}(U_1) = \omega(\eta(U_1), m - h_{\eta(U_1)}(U_1), b[\alpha'_1 \# \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha'_1 \# \alpha_2])$  and  
 $O_{P'}(T^*) = \omega(\eta(T), k' - h_{\eta(T)}(T), a[\omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))])$   
 Since  $\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2]) > \omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))$ ,  $O_{P'}(T^*) <_0 O_P(T)$   
 holds.

(Case 2) The case where cut formula is  $\forall \vec{x} \neg B(\vec{x})$ :

$\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$  is as follows; here  $I_2$  is  $\forall \neg$ -left.

$$\begin{array}{c}
 \frac{S_1^{I_1}}{S_1^{I_1}} I_1 \quad \frac{S_1^{I_2}}{S_1^{I_2}} I_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J}{S_1^J} \quad \frac{S_2^J}{S_2^J} J \\
 \hline
 S^J \\
 \vdots \\
 \overline{T} \quad K \\
 \vdots
 \end{array}
 \quad
 \begin{array}{l}
 S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, \neg B(\vec{x}) \\
 S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^{I_2} : \Pi_1 \rightarrow \Lambda_1, \neg B(\vec{t}) \\
 S_1^{I_2} : \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1 \\
 S_1^J : \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\
 S_2^J : \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda \\
 T : \Phi \rightarrow \Psi
 \end{array}$$

$\langle P', \{h'_\eta\}_{\eta \preceq \xi}, d' \rangle$  is as follows, where  $\tilde{I}_1$  and  $\tilde{I}_2$  are weakening-right and weakening-left (respectively) with weakening formulas  $\forall \vec{x} \neg B(\vec{x})$ . Note that by virtue of (preparation) and (elimination of weakening), any formula of the form  $\neg B(\vec{x})$  which is an ancestor of the auxiliary formula of  $I_1$  is a descendant of principal formulas of an inference  $\neg$ -right. Hence the following  $S_1^{I_1}(\vec{x})$  can be obtained.

$$\begin{array}{c}
 \frac{S_1^{I_1}(t)}{S_1^{I_1}(t)} \tilde{I}_1 \quad \frac{\quad}{\quad} I_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J * (t)}{S_1^J * (t)} \quad \frac{S_2^J}{S_2^J} J \\
 \hline
 S_1^*(t)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\quad}{\quad} I_1 \quad \frac{S_1^{I_2}}{S_1^{I_2}} \tilde{I}_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J}{S_1^J} \quad \frac{S_2^J *}{S_2^J *} J \\
 \hline
 S_2^*
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{\quad}{U_1(t)} K \quad \frac{\quad}{U_2} K}{T^*} N}{\quad} \\
 \vdots
 \end{array}$$

$$\begin{array}{ll}
 S_1^{I_1}(\vec{x}) : & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1 \\
 S_1^{I_1}(\vec{x}) : & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^J * (\vec{x}) : & B(\vec{x}), \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^*(\vec{x}) : & B(\vec{x}), \Gamma, \Pi \rightarrow \Delta, \Lambda \\
 S_2^J * : & \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda, B(\vec{t}) \\
 S_2^* : & \Gamma, \Pi \rightarrow \Delta, \Lambda, B(\vec{t}) \\
 S_1^{I_2} : & \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\
 S_1^{I_2} : & \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\
 U_1(\vec{x}) : & B(\vec{x}), \Phi \rightarrow \Psi \\
 U_2 : & \Phi \rightarrow \Psi, B(\vec{t}) \\
 T^* : & \Phi, \Phi \rightarrow \Psi, \Psi
 \end{array}$$

Since  $lv_{\eta(B(\vec{x}))}(\forall \vec{x} \neg B(\vec{x})) > lv_{\eta(B(\vec{x}))}(B(\vec{x}))$  holds,  $O(P') <_0 O(P)$  is checked as the usual way.

(Case 3) The case where the cut formula of  $J$  is  $Q^{\mathbf{B}}t_s$ :

$\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$  is as follows, where  $K$  (with the lower sequent  $T$ ) denotes the upper most inference below  $J$  such that  $\eta(T) \preceq d(B(X, Q_{\prec t}, t, s)) := i$ ;

Let  $T_1$  denote such upper sequent of  $K$  that is below  $J$ .

$$\begin{array}{c}
 \vdots \\
 \frac{S_1^{I_1}}{S_1^{I_1}} I_1 \quad S \\
 \vdots \\
 \frac{S_1^J}{S_1^J} \quad S_2^J \quad J \\
 \vdots \\
 \frac{T_1 \quad (T_2)}{T} K \\
 \vdots \\
 \vdots
 \end{array}
 \quad
 \begin{array}{l}
 S : \quad t_2 \prec \xi, Q t_2 s_2 \rightarrow B(V, Q_{\prec t_2}, t_2, s_2) \\
 S_1^{I_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_1^{J_1} : \quad \Gamma_1 \rightarrow \Delta_1, Q t_1 s_1 \\
 S_1^J : \quad \Gamma_2 \rightarrow \Delta_2, Q t s \\
 S_2^J : \quad Q t s, \Pi \rightarrow \Lambda \\
 S^J : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2 \\
 T_1 : \quad \Phi_1 \rightarrow \Psi_1 \\
 T_2 : \quad \Phi \rightarrow \Psi
 \end{array}$$

$$O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), c[\omega(\eta(T_1), h_{\eta(T_1)}(T_1), O_P(T_1) \# O_P(T_2))]).$$

$\langle P', \{h'_\eta\}_{\eta \preceq \xi}, d' \rangle$  is as follows, where  $\tilde{I}_1$  is weakening-right with a weakening formula  $Q^B t_1 s_1$ ;

$$\begin{array}{c}
 \vdots \\
 \frac{S_1^{I_1}}{S_1^{I_1}} \tilde{I}_1 \quad S \\
 \vdots \\
 \frac{S_1^{J*}}{S_1^{J*}} \quad S_2^J \quad J \\
 \vdots \\
 \frac{T_1^* \quad (T_2)}{T^*} \\
 \vdots \\
 \frac{T^*}{\tilde{T}^*} \text{sub} \\
 \vdots
 \end{array}
 \quad
 \begin{array}{l}
 S_1^{I_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_1^{J_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1), Q t_1 s_1 \\
 S_1^{J*} : \quad \Gamma_2 \rightarrow \Delta_2, Q t s, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_2^J : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 T_1^* : \quad \Phi_1 \rightarrow \Psi_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 T^* : \quad \Phi \rightarrow \Psi, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 \tilde{T}^* : \quad \Phi \rightarrow \Psi, \mathcal{B}(V, Q_{\prec t_1}, t_1, s_1)
 \end{array}$$

We assign  $\{h'_\eta\}_{\eta \preceq \xi}$  as follows:

- $h'_\eta(\tilde{T}^*) := h_\eta(S)$  for all  $\eta \preceq \xi$
- $h'_\eta(T_1^*) := h_\eta(T_1)$  for all  $\eta \preceq \xi$ .
- $h'_\eta(T^*) := \begin{cases} h_\eta(T) & \text{if } \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{cases}$

$$O_{P'}(\tilde{T}^*) = (i, O_P(T^*))$$

$O_{P'}(T^*) = \omega(\eta(T^*), l - h'_{\eta(T^*)}(T^*), d[\omega(\eta(T_1^*), h_{\eta(T_1^*)}(T_1^*), O_{P'}(T_1^*) \# O_{P'}(T_2^*))])$   
 Note that  $\eta(T^*) = i$ . Obviously  $k = l$  from the figure of  $P'$ . And from the above assignment  $h', c[*] = \omega(\gamma_1, k_1, \dots, \omega(\gamma_s, k_s, d[*]))$  with  $\gamma_s < i = \gamma_{s+1}$ . Hence  $O(P') <_0 O(P)$  holds.

□

The following Corollary is immediate from the above theorem and the fact that  $S_k(ID_\xi^i)$  is a subsystem of  $S_k(Al_\xi^-)$  under the interpretation  $*$  (cf. the paragraph after Definition 7).

**Corollary 1** *The transfinite induction on  $\omega(\xi, k+1, 0)$  is unprovable in  $S_k(ID_\xi^i)$  for  $k > 2$ .*

**Proof.** As remarked after Definition 7,  $\xi$ -level does not change under the interpretation of an  $Al_\xi$ -formula to an  $ID_\xi$ -formula. Hence the Corollary is obvious.

□

**Theorem 2 (Main Theorem)**

$$|S_k(ID_\xi^i(\mathcal{U}_0))| = |S_k(ID_\xi^i)| = |S_k(Al_\xi^-)| = |\omega(\xi, k+1, 0)|_{<_0} \text{ with } k > 2.$$

**Remark 2:** Our system  $S_k(ID_\xi)$  can be reformulated by means of the alternation complexity of quantifiers when we include  $\exists$  in our language. Here, a normal formula is of the form  $Q_1 \bar{x}_1 \bar{Q}_1 \bar{y}_1 \dots Q_n \bar{x}_n \bar{Q}_n \bar{y}_n \forall \bar{y} D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m]$ , where  $D[*_1, \dots, *_m]$  is a context of the language of  $PA$  with no quantifier occurring in  $D$  bounds any  $*_i$  ( $1 \leq i \leq m$ ), and  $\{Q_j, \bar{Q}_j\} = \{\forall, \exists\}$  ( $j = 1, \dots, m$ ).  $lv$  is essentially the same as  $lv_\xi$  except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

$$lv(D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m]) := \begin{cases} 1 & \text{if all } P^{\mathcal{U}} t_i s_i \text{ } (i = 1, \dots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases}$$

Then the  $lv$  of above normal formula is  $n+i$  if  $\bar{Q}_n = \forall$  and  $n+1+i$  if  $\bar{Q}_n = \exists$ , where  $i := lv(D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m])$ .  $S'_k(ID_\xi)$  is defined in the same way as the former definition of  $S_k(ID_\xi)$  with using the above new notation of  $lv$ . It is easily seen that  $S'_k(ID_\xi)$  is equivalent to  $S_k(ID_\xi)$ . In particular  $|S'_k(ID_\xi)| = |\omega(\xi, k+1, 0)|_0$  with  $k > 2$ .

## References

- [1] T. Arai, An Accessibility Proof of Ordinal Diagrams in Intuitionistic Theories for Iterated Inductive Definitions, *Tsukuba J. Math.* 8(1984), 209-218.
- [2] T. Arai, A Subsystem of Classical Analysis Proper to Reduction Method for  $\Pi_1^1$ -Analysis, *Tsukuba J. Math.* 9 (1985), 21-29.
- [3] T. Arai, A Consistency Proof of a System Including Feferman's  $ID_\xi$  by Takeuti's Reduction Method, *Tsukuba J. Math.* 11 (1987), 227-239.
- [4] W. Buchholz, S. Feferman, W. Pohlers and W. Sieg, Iterated Inductive Definitions and Subsystem of Analysis: Recent Proof-Theoretical Studies, *Lecture Notes In Math.* 397. Springer, Berlin (1981)
- [5] W. Buchholz and W. Pohlers, Provable Wellorderings of Formal Theories for Transfinitely Iterated Inductive Definitions, *The Journal of Symbolic Logic* 43 (1978), 118-125.
- [6] S. Feferman, Formal Theories for Transfinite Iteration of Generalized Inductive Definitions and Some Systems of Analysis, *Intuitionism and Proof Theory*, ed. by Kino, Myhill and Vesley, North Holland, (1970).

- [7] G. Gentzen, Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie, *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften . Neue Folge* 4 (1938), 19-44.
- [8] G. Gentzen. Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie, *Mathematische Annalen*, 119 (1943), 140-161
- [9] G. E. Mints, Quantifier-free and One-quantifier Systems, *Zap. Nauch. Sem., LOMI Stek. Akad. Nauk SSSR* 20 (1971), 115-133.
- [10] G. E. Mints, Exact Estimates of the Provability of Transfinite Induction in the Initial Segment of Arithmetic, *Zap. Nauch. Sem., LOMI Stek. Akad. Nauk SSSR* 20 (1971), 134-144.
- [11] W. Pohlers, Ordinals Connected with Formal Theories for Transfinitely Iterated Inductive Definitions, *J. Symbolic Logic* 43 (1978), 161-182.
- [12] T. Shimura, A Weakened Version of Arai's Theory  $AI_{\xi}^-$  and Its Proof Theoretic Ordinal, *unpublished*
- [13] K. Shirai, A Relation between Transfinite Induction and Mathematical Induction in Elementary Number Theory, *Tsukuba J. Math.*, 1 (1977), 91-124.
- [14] G. Takeuti, *Proof Theory*, 2nd edition, North Holland, 1987.